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LOGISTIC MAP NEURAL MODELS

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Abstract. This paper examines the ability of neural based structures to model the logistic equation. This modelling includes not only the generation of the logistic curve, but also the time series that are generated by the logistic neural model. This study concerns all main regions of the logistic equation: the region of convergence for parameter values less than 3, the periodic region for parameter values in the interval [3, 3.57], and the chaotic region for values in the interval [3.57, 4]. For each region, the fixed points of the logistic map are calculated and compared to the corresponding theoretical points, followed by an analysis of the distribution of the absolute mean error between the theoretical and the experimental curves. Finally, the Lyapunov exponent and the fractal dimensions for both the theoretical and the neural based attractor are estimated.

1 INTRODUCTION

The logistic map is one of the most interesting chaotic systems, and the study of its behavior is a powerful tool for the understanding of their main features. These features include sensitivity to the initial conditions, and a positive Lyapunov exponent as it is calculated by the chaotic time series emerging from the time evolution of the chaotic system. The transition from determinism to chaos evolves through a period doubling mechanism, in which, for specific parameter values, the number of the fixed periodic points is doubled. The mathematical equation that describes the logistic map, has the form

$$x_{n+1} = \lambda x_n (1 - x_n) \tag{1}$$

Models for the logistic map can be developed in a variety of ways. In this paper we restrict ourselves to the neural based models, which are implemented by means of neural network structures. There are two main reasons that justify the usage of such models [1]. The first reason is the capability of neural networks to simulate nonlinear mappings, and the second reason is their ability to process unknown input values that do not belong to the training set. This property of neural networks is known as generalization and allows the estimation of an output for an unknown input, using interpolation techniques. Since the logistic

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map is an example of nonlinear mapping, neural network structures are the most appropriate structures for the implementation of logistic map simulation models.

The main advantage of creating such models is the fact that we can use these models to generate artificially created chaotic time series, in the same way that they are generated from the theoretical model. This allows us to study the ability of artificial models to generate chaotic data that can be used in many applications such as chaos control.

2 OVERVIEW OF THE CURRENT WORK

This research examines several structures for the neural models. The best one, in terms of accuracy, was selected to simulate the behavior of the logistic map, for certain λ values. These values belong to the three regions of the logistic map, that is, the region of convergence, the periodic region, and the chaotic region. For the region of convergence, we model the logistic map for λ equal to 2.50 and 3.00. For the periodic region, these values are 3.10 (2-period trajectory) and 3.50 (4-period trajectory). For the chaotic region we restrict ourselves to the values 3.90, 3.93 and 3.96.

For each case, the logistic curve is reconstructed by means of the neural network, and the absolute mean error between the theoretical and the neural based logistic curve, is calculated. In order to characterize the accuracy of this reconstruction, the distribution of the absolute mean error, i.e. the percentage of data points for which the calculated error belongs to specific intervals, is used. In most cases, the 50% of the data points gave an error less than 0.001, while a percentage equal to 80% gave an error less than 0.002. Besides this logistic curve reconstruction, another characteristic taken into account was the creation of the time series from the logistic map neural model. The procedure that gives these time series is the same as in the theoretical model. Starting from a specific initial condition, the neural network generates an experimental time series using a feedback technique - this means that the n_{th} output of the networks is used as input in order to generate the $(n+1)_{th}$ output. This procedure is described in Figure 2.

The study of the features associated with this experimental time series is the second task performed in this research. For each λ value, the time evolution of the experimental time series must be the same, as in the theoretical case. This similarity includes the same periodicity, the same value for the fixed points, and the same Lyapunov exponent and fractal dimensions. So, for each experimental time series, the time evolution of the series is studied, and the values of the experimental fixed points and the experimental Lyapunov exponent are calculated. This calculation is also performed for the values of the capacity, information, and correlation dimension associated with the logistic map chaotic attractor. The comparison between the theoretical and the experimental values is used to justify the accuracy and the quality of the logistic map neural model.

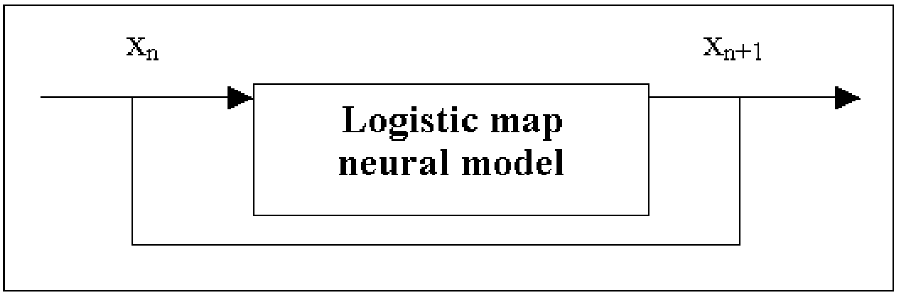

Fig. 1. Time series generation from the logistic map neural model

3 NEURAL NETWORK ARCHITECTURE

The neural network used for the modelling of the logistic equation is a three layered feed forward neural network trained through the back propagation algorithm [3]. The network structure includes one input neuron, three hidden neurons, and one output neuron [2] as shown in Figure 2. In the input layer there are four additional functional link neurons [4], with an output of the form $sin(kpx)$ where x is the output of the input neuron, and the parameter k gets the values 1,2,3 and 4. The network is a fully interconnected neural network and all the neurons, except those of the input layer, get an additional input from a bias unit with a fixed value equal to -1. This allows the usage of a variable threshold for these neurons. The activation function for all neurons was the sigmoidal function, with a sigmoidal slope equal to 1. The optimum parameter values for the back propagation training were found to be 0.9 for the learning rate, and 0.4 for the momentum. In some cases, we tried to use variable learning rate but the results were not very different - so, the learning rate has a fixed value. Regarding the number of training circles, it was between 50000 and 350000, with a mean value equal to 100000 iterations. Finally, the training mode used was the online training. Thus, the update of the synapse weights was performed after the processing of every training pattern.

The training set that the neural network had to learn included 1000 pairs in the form (x, y), where the x values were uniformly distributed in the interval [0, 1], while the y values, were calculated by the equation $y = \lambda x(1-x)$. Initially, the network was trained using 500 data pairs, but later on, the training procedure was repeated, using an 1000-point training set. This decision was based on the fact, that the denser the x points, the more accurate the neural based y points. As a result, a data set of 1000 data pairs was used in all cases.

The recall procedure included the recall of the above 1000 points, for the λ value for which the network was trained, and for other λ values close to that. More specifically, for each λ value for which a neural model was constructed, the recall procedure performed for values belonging to the interval $[\lambda - 0.5, \lambda + 0.5]$.

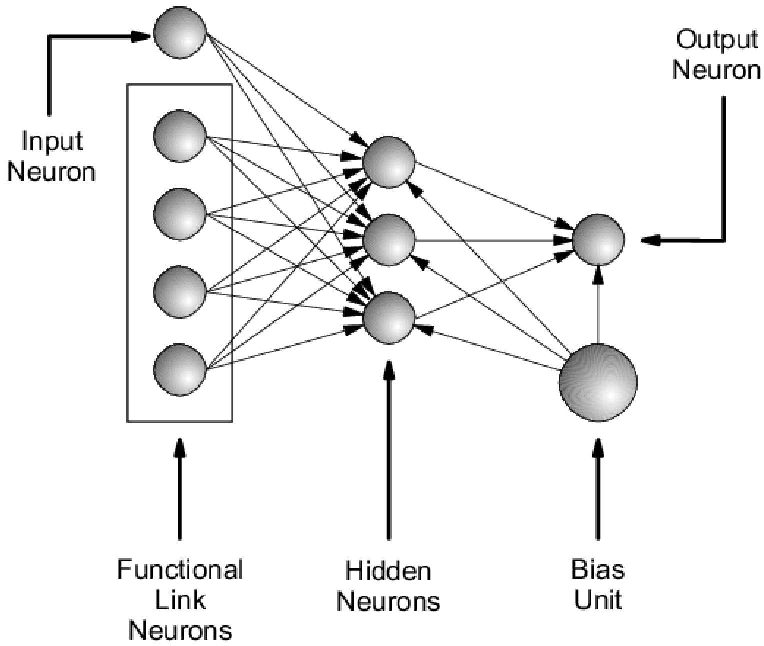

Fig. 2. The selected neural network structure

4 NEURAL MODELS FOR THE REGION OF CONVERGENCE

When the λ parameter gets values less than 3, the time series generated by applying the recursive equation (1), converges to a single fixed point. The larger the value of λ parameter, the larger the value of the fixed point and the number of iterations needed to reach it. In the special case for which $\lambda = 3$, the fixed point has the value 0.6666667 and the theoretical time series converges to it, after many millions of iterations. However the neural based model could not behave in this way - since it is only an approximation of the real system - so, the experimental time series converges to the corresponding experimental fixed point, after a few thousands of iterations. It should be noted that due to being a special case, there is a large difference between the experimental and the theoretical fixed points, for $\lambda = 3$. Table 1 contains the theoretical and the experimental fixed points, both with the number of iterations needed in order to reach them, for the λ values 2.50 and 3.00. In all cases the theoretical and experimental time series, are produced using an initial condition equal to 0.1

From the above description, it can be seen that the experimental time series generated by the neural model, behaves in the same way as in the theoretical case, in the sense that it is characterized by the same time evolution and converges to the experimental fixed points, which are very close to the theoretical points. Another interesting feature of these neural models is the distribution of the

Table 1. Theoretical and experimental fixed points for the region of convergence

λ Value	Theoretical Fixed Point	Experimental Fixed Point	Iterations for Theoretical Point	Iterations for Experimental Fixed Point
2.50	0.600000	0.599853	19	29
3.00	0.666666	0.646801	> 5000000	1372

absolute mean error between the theoretical and the experimental logistic curves. Table 2 includes this distribution for each neural model of the logistic function.

Table 2. Absolute mean error distribution between the theoretical and the neural based curve for the region of convergence

Error Intervals		Points	%	Points	%
		$\lambda = 2.50$		$\lambda = 3.00$	
< 0.001		778	77.8	830	83.0
0.001	0.002	167	16.7	128	12.8
0.002	0.003	038	03.8	017	01.7
0.003	0.004	005	0.50	010	01.0
0.004	0.005	002	0.20	009	0.90
> 0.005		010	01.0	006	0.60

Table 3 contains the absolute mean error between the theoretical and the recalled logistic curves, for λ values close to the value $\lambda_{trained}$ for which the neural model had been created. These values belong to the interval [$\lambda_{trained}$-0.5, $\lambda_{trained}$+0.5].

5 NEURAL MODELS FOR THE PERIODIC REGION

When the λ parameter gets values in the interval [3, 3.57], the generated time series is a periodic one, with the number of points in each period, depending on the value of λ. Two cases are examined here, corresponding to λ values that lead to period doubling. These values are 3.10 (2-period orbit) and 3.50 (4-period orbit). As in the previous cases, the calculation includes an estimation of the experimental periodic points for each case, the absolute mean error distribution, and the accuracy of the logistic curve recall, for λ values close to $\lambda_{trained}$. These results are represented in the Tables 4, 5 and 6.

Table 3. Absolute mean error distribution between the theoretical and the neural based curve for the region of convergence and for λ close to $\lambda_{trained}$

$\lambda_{trained} = 2.50$		$\lambda_{trained} = 2.50$	
λ Value	Error	λ Value	Error
2.45	0.008376	2.95	0.008433
2.46	0.006725	2.96	0.006779
2.47	0.005076	2.97	0.005126
2.48	0.003435	2.98	0.003474
2.49	0.001893	2.99	0.001886
2.50	**0.000739**	**3.00**	**0.000741**
2.51	0.001955	3.01	0.001897
2.52	0.003579	3.02	0.003519
2.53	0.005230	3.03	0.005155
2.54	0.006894	3.04	0.006806
2.55	0.008559	3.05	0.008433

Table 4. Theoretical and experimental fixed points for the periodic region

λ Value	Theoretical Fixed Point	Experimental Fixed Point	Iterations for Theoretical Point	Iterations for Experimental Fixed Point
Period 2 region				
3.10	0.558014	0.553057	56	48
	0.764567	0.767024	65	45
Period 4 region				
3.50	0.826941	0.827224	36	44
	0.500884	0.501112	37	45
	0.874997	0.875502	38	46
	0.382820	0.383010	39	47

Table 5. Absolute mean error distribution between the theoretical and the neural based curve for the periodic region

Error Intervals		Points	%	Points	%
		$\lambda = 3.10$		$\lambda = 3.50$	
< 0.001		654	65.4	629	62.9
0.001	0.002	303	30.3	311	31.1
0.002	0.003	036	03.6	037	03.7
0.003	0.004	001	00.1	013	01.3
0.004	0.005	001	00.1	001	00.1
> 0.005		005	00.5	009	00.9

Table 6. Absolute mean error distribution between the theoretical and the neural based curve for the periodic region and for λ close to $\lambda_{trained}$

$\lambda_{trained} = 3.10$		$\lambda_{trained} = 3.50$	
λ Value	Error	λ Value	Error
3.05	0.008436	3.45	0.008416
3.06	0.006777	3.46	0.006762
3.07	0.005119	3.47	0.005108
3.08	0.003462	3.48	0.003478
3.09	0.001876	3.49	0.001932
3.10	**0.000876**	**3.50**	**0.000958**
3.11	0.001794	3.51	0.002016
3.12	0.003403	3.52	0.003586
3.13	0.005051	3.53	0.005218
3.14	0.006707	3.54	0.006864
3.15	0.008371	3.55	0.008522

6 NEURAL MODELS FOR THE CHAOTIC REGION

The last case concerns the development of neural models for λ values that belong to the chaotic region. Logistic map neural models have been created for the λ values 3.90, 3.93 and 3.96. These values cover a significant part of the chaotic region which has an upper bound that corresponds to the value $\lambda = 4.00$. The experimental results produced by these models include the distribution of the absolute mean error between the theoretical and the neural based logistic curve, and the recall results for λ values close to the value $\lambda_{trained}$ for which the neural model has been constructed. These results are shown in Tables 7 and 8.

Table 7. Absolute mean error distribution between the theoretical and the neural based curve for the chaotic region

Error Intervals		Points	%	Points	%	Points	%
		$\lambda = 3.90$		$\lambda = 3.93$		$\lambda = 3.96$	
< 0.001		814	81.4	680	68.0	660	66.0
0.001	0.002	163	16.3	268	26.8	296	29.6
0.002	0.003	018	01.8	047	04.7	038	03.8
0.003	0.004	002	00.2	002	00.2	004	00.4
0.004	0.005	001	00.1	000	00.0	000	00.0
> 0.005		002	00.2	003	00.3	002	00.2

Table 8. Absolute mean error distribution between the theoretical and the neural based curve for the chaotic region and for λ close to $\lambda_{trained}$

$\lambda_{trained} = 3.90$		$\lambda_{trained} = 3.93$		$\lambda_{trained} = 3.96$	
λ Value	Error	λ Value	Error	λ Value	Error
3.85	0.008408	3.88	0.008404	3.91	0.008383
3.86	0.006745	3.89	0.006741	3.92	0.006720
3.87	0.005090	3.90	0.005093	3.93	0.005058
3.88	0.003453	3.91	0.003458	3.94	0.003406
3.89	0.001828	3.92	0.001848	3.95	0.001784
3.90	**0.000595**	**3.93**	**0.000761**	3.96	**0.000842**
3.91	0.001789	3.94	0.001830	3.97	0.001808
3.92	0.003392	3.95	0.003404	3.98	0.003396
3.93	0.005046	3.96	0.005049	3.99	0.005052
3.94	0.006704	3.97	0.006707	4.00	0.006712
3.95	0.008362	3.98	0.008366	—	—

7 LYAPUNOV EXPONENT AND FRACTAL DIMENSION ESTIMATION

A good criterion for the characterization of the logistic map neural model is the estimation of the Lyapunov exponent for the experimental time series produced by the neural model, and its comparison to the corresponding theoretical Lyapunov exponent calculated from the theoretical time series. It can be proven that for the special case of the logistic map, the Lyapunov exponent is given by the equation [5]:

$$\alpha = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} log_2 |\lambda(1 - 2x_i)| \tag{2}$$

where N is the number of data points and λ the parameter value of the logistic map equation. Since we have the ability to create both time series - one theoretical and one experimental - for each λ value, we can estimate the theoretical and the experimental Lyapunov exponents by applying the above equation, and thus, evaluate the accuracy of the modeling of the logistic map. The calculation of the Lyapunov exponent was performed for each λ value for which we have created a neural model, and the results of this calculation are shown in Table 9.

Figure 3 shows the variation of the Lyapunov exponent for various values of λ parameter, for both the theoretical and the experimental cases.

The other characteristic feature calculated for both the theoretical and the experimental time series was the values of the three main dimension types associated with each chaotic system, that is, the capacity, the information, and the correlation dimension.

Table 9. Theoretical and experimental Lyapunov exponent for all the neural models created for the region of convergence, the periodic rerion, and the chaotic region

λ Value	Theoretical Lyapunov exponent	Experimental Lyapunov exponent
2.50	- 0.970689	- 0.972358
3.00	- 0.035070	- 0.042604
3.10	- 0.308960	- 0.352512
3.50	- 0.903258	- 0.759712
2.90	+ 0.723355	+ 0.758118
3.93	+ 0.881630	+ 0.806589
3.96	+ 0.905534	+ 0.866076

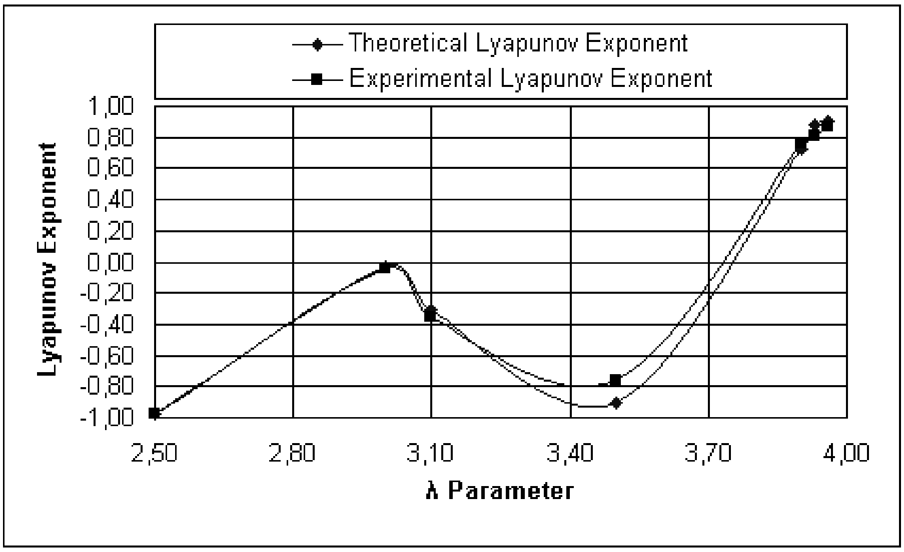

Fig. 3. Variation of the Lyapunov exponent as a function of λ parameter for the theoretical and the experimental cases

The capacity dimension is given by the equation [5]

$$d_{cap} = \lim_{\varepsilon \to 0} \frac{ln(N(\varepsilon))}{ln(1/\varepsilon)} \tag{3}$$

where $(N(\varepsilon)$ is the number of hyper-cubes of side size ε that are required in order to cover the attractor's shape. The information dimension is defined as [6]

$$d_{inf} = \lim_{l \to 0} \frac{-S(l)}{log(l)} \tag{4}$$

where $S(l)$ is the minimum information that is required, in order to specify a point in a set S with an accuracy of l. Finally, the correlation dimension is calculated by the equation

$$d_{cor} = \lim_{r \to 0} \frac{log(C(r))}{log(r)} \tag{5}$$

where C(r) is the correlation function that gives the percentage of points of a system trajectory, whose distance is less than r.

The computation of the capacity, information and correlation dimensions for both the theoretical and the experimental time series is performed through a software package developed by Sarraille and DiFalco [7], named FD3. Table 10 includes the calculated values for the capacity, information, and correlation dimension for the theoretical curve, as well as the neural based curve. From the above table it is clear that the agreement between the theoretical and the experimental results is sufficient, especially in the case of large λ values, where the system time series are extremely chaotic.

Table 10. Capacity, information and correlation dimension for the theoretical and the neural based time series associated with the logistic map, for various λ values

λ	Theoretical dimension			Experimental dimension		
	d_{cap}	d_{inf}	d_{cor}	d_{cap}	d_{inf}	d_{cor}
2.50	0.26149	0.24854	0.16990	0.26149	0.24049	0.15958
3.00	0.68895	0.77769	0.73209	0.60443	0.57141	0.43861
3.10	0.44752	0.34209	0.20971	0.48580	0.43798	0.30538
3.50	0.60741	0.57348	0.48487	0.57846	0.61243	0.56844
3.90	0.97710	0.94312	0.89089	0.97383	0.93844	0.88953
3.93	0.97947	0.94662	0.89629	0.97879	0.94289	0.88655
3.96	0.97657	0.92181	0.79947	0.97581	0.93453	0.86197

Figures 4, 5 and 6 show the variation of the capacity, information and correlation dimension as a function of λ parameter, for both the theoretical and the experimental cases.

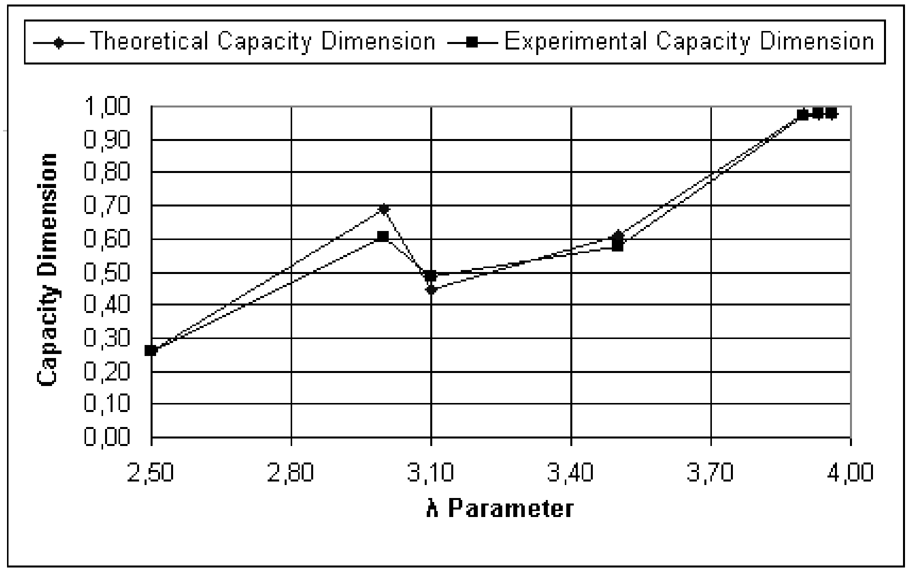

Fig. 4. Variation of the capacity dimension as a function of λ parameter

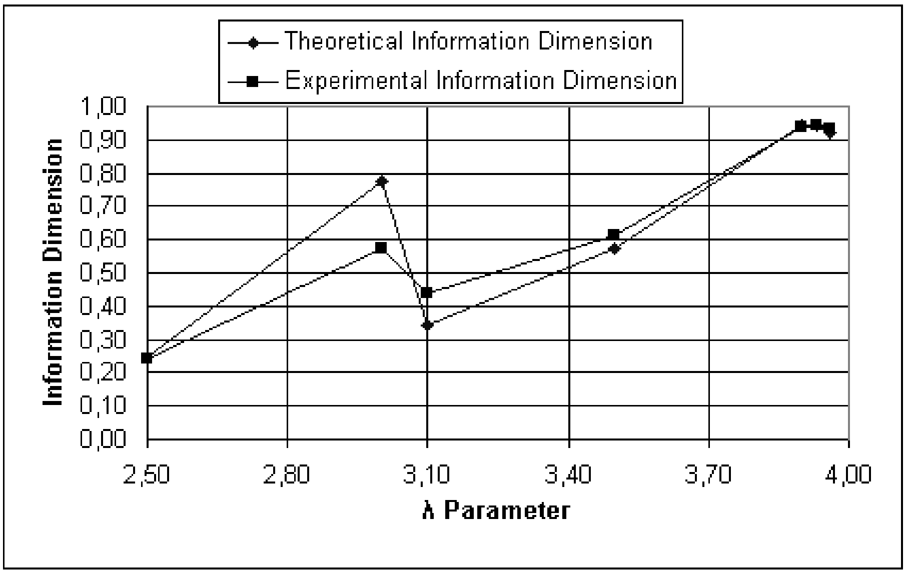

Fig. 5. Variation of the information dimension as a function of λ parameter

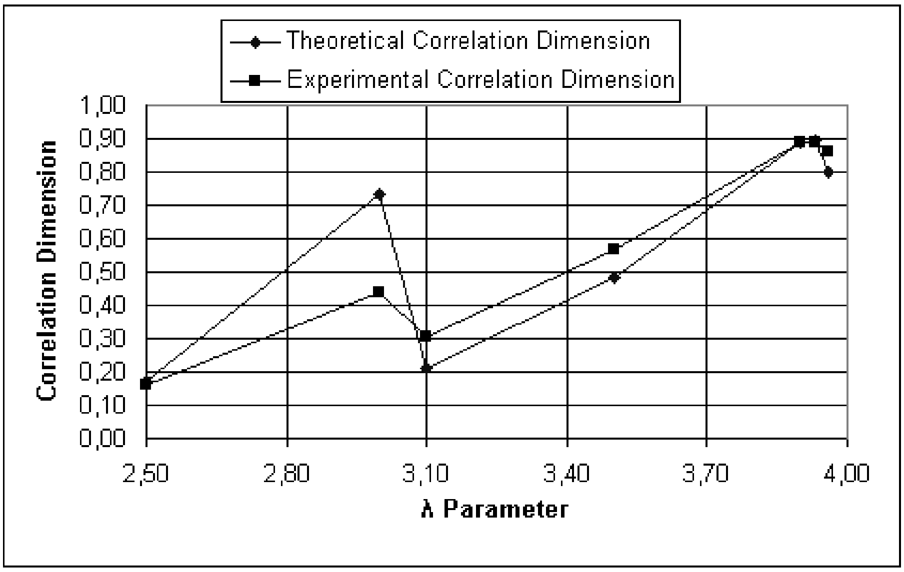

Fig. 6. Variation of the correlation dimension as a function of λ parameter

8 CONCLUSIONS

The aim of this research was the development of neural models for the logistic equation - for selected λ values - and the investigation of the features associated with the time series generated by these models. In all cases, the behavior of these models matches very closely the actual theoretical model. This result is valid for the type of periodicity and the values of the periodic fixed points, as well as the value and the sign of the Lyapunov exponent and fractal dimensions for each value. However, since this model is only an approximation of the actual system, it is not possible to behave exactly in the same way in some very special cases. So, for the value $\lambda = 3$, where the fixed point is reached after many millions of iterations, the neural based time series reaches this point after only about 1400 iterations.

Another very important conclusion is that, in the chaotic region, the neural based time series is very different from the corresponding theoretical time series, even though the absolute mean square error between the theoretical and the experimental logistic curve is very small - for $\lambda = 3.90$ this error is equal to 0.000595 which is the best calculated error value. This deviation between the two series is caused by the properties of the logistic map in the chaotic region. This small error value is a mean error value, which means that there are some points for which this error is large enough. On the other hand, in the chaotic region of the logistic map, it is obvious that after many iterations, the time series curve - cobweb plot - will pass from every point of the logistic curve. If in some iteration the curve passes from a point that leads to a large error value, the next point will be quite different from the corresponding theoretical point and after a large number of iterations, the two series will be quite different from each other. However, a positive Lyapunov exponent characterizes both series so they are chaotic, while, in the same way, the theoretical and the experimentally calculated fractal dimensions have very close values. In other words, the neural models created for the logistic map give exactly the same results as in the theoretical case, for the region of convergence and the periodic region, but in the chaotic region, the agreement is only qualitative, not quantitative.

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