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Composition Algorithms for Cardinal Direction Relations

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Abstract. We present a formal model for qualitative spatial reasoning with cardinal directions that is based on a recent proposal in the literature. We use our formal framework to study the composition operation for the cardinal direction relations of this model. We consider progressively more expressive classes of cardinal direction relations and give composition algorithms for these classes. Finally, when we consider the problem in its generality, we show that the binary relation resulting from the composition of some cardinal direction relations cannot even be expressed using the relations which are currently employed by the related proposal.

1 Introduction

The composition operator has received a lot of attention in the area of qualitative spatial reasoning [7, 15, 5]. and has been studied for several kinds of useful spatial relations like topological relations [2, 3, 15], direction relations [6, 10] and qualitative distance relations [5, 4]. Typically, the composition operator is used as a mechanism for inferring new relations from existing ones. Such inference mechanisms are very important as they are in the heart of any system that retrieves collections of objects similarly related to each other using spatial relations [13]. Moreover, composition is used to identify classes of relations that have a tractable consistency problem [7, 15, 12].

This work concentrates on qualitative spatial reasoning with *cardinal direction relations* [6, 10]. Cardinal direction relations describe how regions of space are placed relative to one another (e.g., region a is *north of* region b). We study the recent model of Goyal and Egenhofer [6]. This model is currently one of the most expressive models for qualitative reasoning with cardinal directions. It works with extended regions and has potential in Multimedia and Geographic Information Systems applications [3, 4].

In this paper, we give formal definitions for the cardinal direction relations that can be expressed in the model of Goyal and Egenhofer [6]. Then, we use our formal framework to study the composition operation for cardinal direction relations in the model of [6]. Goyal and Egenhofer first studied this operation in [6] but their method does not always work correctly. The previous observation leaves us with the task of finding a correct method for computing the composition. To do this, we consider progressively more expressive classes of cardinal direction relations and give composition algorithms for these classes. Finally, when we consider the problem in its generality, we show that the binary relation resulting from the composition of some cardinal direction relations *cannot even be expressed* using the relations defined in [6]. A more detailed discussion as well as the proofs of the above results appear in [16].

The rest of the paper is organized as follows. Section 2 presents the formal model. In Sections 3 and 4 we consider two subclasses of cardinal direction relations and give composition algorithms for these classes. In Section 5 we show that the result of the composition of some cardinal direction relations cannot be expressed using the relations defined in [6]. Our conclusions are presented in Section 6.

2 A Formal Model for Cardinal Direction Information

We consider the Euclidean space $\Re^2$. *Regions* are defined as non-empty and bounded sets of points in $\Re^2$. Let a be a region. The *greatest lower bound* or the *infimum* [11] of the *projection* of region a on the x-axis (respectively y-axis) is denoted by $inf_x(a)$ (respectively $inf_y(a)$). The *least upper bound* or the *supremum* of the *projection* of region a on the x-axis (respectively y-axis) is denoted by $sup_x(a)$ (respectively $sup_y(a)$). We will often refer to sup and inf as *endpoints*.

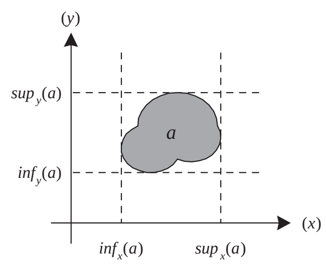

Fig. 1. A region and its bounding box

The *minimum bounding box* of a region a, denoted by $mbb(a)$, is the box formed by the straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$ and $y =$

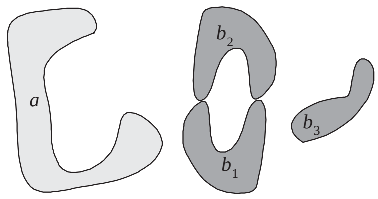

Fig. 2. Regions

$sup_y(a)$ (see Figure 1). Obviously, the projections on the x-axis (respectively y-axis) of a region and its minimum bounding box have the same endpoints.

We will consider throughout the paper the following types of regions:

– Regions that are homeomorphic to the *unit disk* [11]. The set of these regions will be denoted by REG. Regions in REG are *closed*, *connected* and have *connected boundaries* (for definitions see [1, 11])
– Regions that are formed by finite unions of regions in REG. The set of these regions will be denoted by REG^*. Notice that regions in REG^* can be *disconnected* and can have *holes*.

In Figure 2, regions a, b_1, b_2 and b_3 are in REG (also in REG^*) and region $b = b_1 \cup b_2 \cup b_3$ is in REG^*.

Regions in REG have been previously studied in [6, 14]. They can be used to model areas in various interesting applications, e.g., land parcels in Geographic Information Systems [3, 4]. In the sequel we will formally define cardinal direction relations for regions in REG. To this end, we will need regions in REG^*.

Let us now consider two arbitrary regions a and b in REG. Let region a be related to region b through a cardinal direction relation (e.g., a is north of b). Region b will be called the *reference* region (i.e., the region *to* which the relation is described) while region a will be called the *primary* region (i.e., the region *from* which the relation is described) [6]. The axes forming the minimum bounding box of the reference region b divide the space into 9 tiles (Figure 3a). The peripheral tiles correspond to the eight cardinal direction relations *south*, *southwest*, *west*, *northwest*, *north*, *northeast*, *east* and *southeast*. These tiles will be denoted by $S(b)$, $SW(b)$, $W(b)$, $NW(b)$, $N(b)$, $NE(b)$, $E(b)$ and $SE(b)$ respectively. The central area corresponds to the region's minimum bounding box and is denoted by $B(b)$. By definition each one of these tiles includes the parts of the axes forming it. The union of all 9 tiles is $\Re^2$.

If a primary region a is included (in the set-theoretic sense) in tile $S(b)$ of some reference region b (Figure 3b) then we say that *a is south of b* and we write $a\ S\ b$. Similarly, we can define *southwest* (SW), *west* (W), *northwest* (NW), *north* (N), *northeast* (NE), *east* (E), *southeast* (SE) and *bounding box* (B) relations.

If a primary region a lies partly in the area $NE(b)$ and partly in the area $E(b)$ of some reference region b (Figure 3c) then we say that *a is partly northeast and partly east of b* and we write $a\ NE{:}E\ b$.

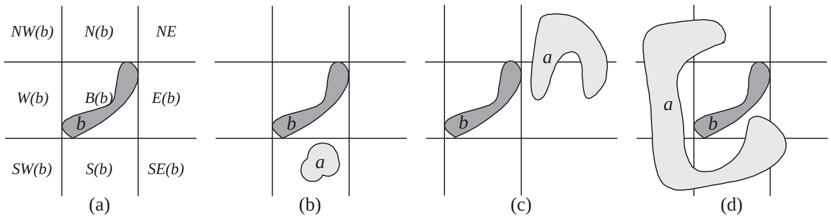

Fig. 3. Reference tiles and relations

The general definition of a cardinal direction relation is as follows.

Definition 1. *An* atomic cardinal direction relation *is an element of the set* $\{B, S, SW, W, NW, N, NE, E, SE\}$. *A* basic cardinal direction relation *is an atomic cardinal direction relation or an expression* $R_1:\cdots:R_k$ *where* $2 \leq k \leq 9$, $R_1, \ldots, R_k \in \{B, S, SW, W, NW, N, NE, E, SE\}$, $R_i \neq R_j$ *for every* i, j *such that* $1 \leq i, j \leq k$ *and* $i \neq j$, *and the tiles* $R_1(b), \ldots, R_k(b)$ *form a region of* REG *for any reference region* b.

Example 1. The following are basic cardinal direction relations:

$$S, \quad NE{:}E \text{ and } B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE.$$

Regions involved in these relations are shown in Figures 3b, 3c and 3d respectively.

In order to avoid confusion we will write the atomic elements of a cardinal direction relation according to the following order: B, S, SW, W, NW, N, NE, E and SE. Thus, we always write $B{:}S{:}W$ instead of $W{:}B{:}S$ or $S{:}B{:}W$. The readers should also be aware that for a basic relation such as $B{:}S{:}W$ we will often refer to B, S and W as its *tiles*.

2.1 Defining Basic Cardinal Direction Relations Formally

Now we can formally define the atomic cardinal direction relations B, S, SW, W, NW, N, NE, E and SE of the model as follows:

- $a\ B\ b$ iff $inf_x(b) \leq inf_x(a)$, $sup_x(a) \leq sup_x(b)$, $inf_y(b) \leq inf_y(a)$ and $sup_y(a) \leq sup_y(b)$.
- $a\ S\ b$ iff $sup_y(a) \leq inf_y(b)$, $inf_x(b) \leq inf_x(a)$ and $sup_x(a) \leq sup_x(b)$.
- $a\ SW\ b$ iff $sup_x(a) \leq inf_x(b)$ and $sup_y(a) \leq inf_y(b)$.
- $a\ W\ b$ iff $sup_x(a) \leq inf_x(b)$, $inf_y(b) \leq inf_y(a)$ and $sup_y(a) \leq sup_y(b)$.
- $a\ NW\ b$ iff $sup_x(a) \leq inf_x(b)$ and $sup_y(b) \leq inf_y(a)$.
- $a\ N\ b$ iff $sup_y(b) \leq inf_y(a)$, $inf_x(b) \leq inf_x(a)$ and $sup_x(a) \leq sup_x(b)$.
- $a\ NE\ b$ iff $sup_x(b) \leq inf_x(a)$ and $sup_y(b) \leq inf_y(a)$.
- $a\ E\ b$ iff $sup_x(b) \leq inf_x(a)$, $inf_y(b) \leq inf_y(a)$ and $sup_y(a) \leq sup_y(b)$.
- $a\ SE\ b$ iff $sup_x(b) \leq inf_x(a)$ and $sup_y(a) \leq inf_y(b)$.

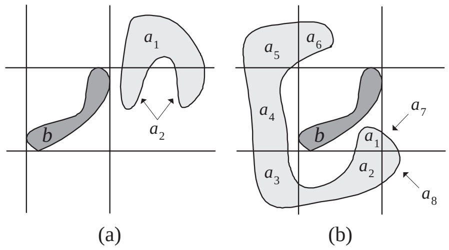

Fig. 4. Relations and component variables

Using the above atomic relations we can define all non-atomic ones. For instance relation $NE{:}E$ (Figure 4a) and relation $B{:}S{:}SW{:}W{:}NW{:}N{:}E{:}SE$ (Figure 4b) are defined as follows:

a $NE{:}E$ b iff there exist regions a_1 and a_2 in REG^* such that $a = a_1 \cup a_2$, a_1 NE b and a_2 E b.

a $B{:}S{:}SW{:}W{:}NW{:}N{:}SE{:}E$ b iff there exist regions $a_1, \ldots, a_8$ in REG^* such that $a = a_1 \cup a_2 \cup a_3 \cup a_4 \cup a_5 \cup a_6 \cup a_7 \cup a_8$, a_1 B b, a_2 S b, a_3 SW b, a_4 W b, a_5 NW b, a_6 N b, a_7 SE b and a_8 E b.

In general each non-atomic cardinal direction relation is defined as follows. If $2 \leq k \leq 9$ then

a $R_1{:}\cdots{:}R_k$ b iff there exist regions $a_1, \ldots, a_k \in REG^*$ such that $a = a_1 \cup \cdots \cup a_k$, a_1 R_1 b, a_2 R_2 b, $\ldots$, and a_k R_k b.

The variables $a_1, \ldots, a_k$ in any equivalence such as the above are in general in REG^*. For instance let us consider Figure 4a. The lines forming the bounding box of the reference region b divide region $a \in REG$ into two components a_1 and a_2. Clearly a_2 is in REG^* but not in REG. Notice also that for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$, a_i and a_j have disjoint interiors but may share points in their boundaries.

Each of the above cardinal direction relations can also be defined using set-theoretic notation as *binary relations* consisting of all pairs of regions satisfying the right-hand sides of the "iff" definitions. The reader should keep this in mind throughout the paper; this equivalent way of defining cardinal direction relations will be very useful in Section 5.

The set of basic cardinal direction relations in this model contains 218 elements. We will use $\mathcal{D}$ to denote this set. Relations in $\mathcal{D}$ are jointly exclusive and pairwise disjoint. Elements of $\mathcal{D}$ can be used to represent *definite information* about cardinal directions, e.g., a N b. An enumeration and a pictorial representation for all relations in $\mathcal{D}$ can be found in [6].

Using the 218 relations of $\mathcal{D}$ as our basis, we can define the *powerset* $2^{\mathcal{D}}$ of $\mathcal{D}$ which contains 2^{218} relations. Elements of $2^{\mathcal{D}}$ are called *cardinal direction* *relations* and can be used to represent not only definite but also *indefinite information* about cardinal directions e.g., $a\ \{N,\ W\}\ b$ denotes that region a is north *or* west of region b. [6] considers only a small subset of the disjunctive relations of $2^{\mathcal{D}}$ through a nice pictorial representation called the *direction-relation* matrix.

In the following sections we will study the operation of composition for cardinal direction relations. Let us first define the composition operation for arbitrary binary relations [11].

Definition 2. *Let R_1 and R_2 be binary relations. The* composition *of relations R_1 and R_2, denoted by $R_1 \circ R_2$, is another binary relation which satisfies the following. For arbitrary regions a and c, $a\ R_1 \circ R_2\ c$ holds if and only if there exists a region b such that $a\ R_1\ b$ and $b\ R_2\ c$ hold.*

A *composition table* stores the result of the composition $R_1 \circ R_2$ for every pair of relations R_1 and R_2. For the case of cardinal direction relations, the composition table has $218^2 = 47524$ entries. Therefore, since it is rather painful to manually calculate each entry, one has to develop appropriate algorithms to for the calculation of composition. Goyal and Egenhofer have proposed a method for composing cardinal direction relations in [6]. Unfortunately, the method presented in [6] does not calculate the correct composition for several cases (see [16] for details). To remedy this our approach addresses the composition problem one step at a time. In the next section we will consider the simplest case, i.e., the composition of an atomic with a basic (atomic or non-atomic) cardinal direction relation.

3 Composing an Atomic with a Basic Cardinal Direction Relation

In Figure 5 we show the composition table for atomic cardinal direction relations [6]. The table uses the function symbol δ as a shortcut. For arbitrary atomic cardinal direction relations $R_1, \ldots, R_k$, the notation $\delta(R_1, \ldots, R_k)$ is a shortcut for the disjunction of all valid *basic* cardinal direction relations that can be constructed by combining atomic relations $R_1, \ldots, R_k$. For instance, $\delta(SW, W, NW)$ stands for the disjunctive relation:

$$\{SW,\ W,\ NW,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

Moreover, we define:

$$\delta(\ \delta(R_{11}, \ldots, R_{1k_1}), \delta(R_{21}, \ldots, R_{2k_2}), \ldots, \delta(R_{m1}, \ldots, R_{mk_m})\ ) = \delta(R_{11}, \ldots, R_{1k_1}, R_{21}, \ldots, R_{2k_2}, \ldots, R_{m1}, \ldots, R_{mk_m}).$$

Application of the operator δ as it has been defined, suffices for our needs in this paper.

As usual U_{dir} stands for the universal cardinal direction relation. The correctness of the transitivity table can easily be verified using the definitions of Section 2 and the definition of composition (Definition 2).

$R_1 \backslash R_2$	N	NE	E	SE
N	N	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$
NE	$\delta(N, NE)$	NE	$\delta(NE, E)$	$\delta(NE, E, SE)$
E	$\delta(N, NE)$	NE	E	SE
SE	$\delta(N, NE, E, SE, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE
S	$\delta(N, S, B)$	$\delta(NE, E, SE)$	$\delta(E, SE)$	SE
SW	$\delta(S, SW, W, NW, N, B)$	U_{dir}	$\delta(E, SE, S, SW, W, B)$	$\delta(SE, S, SW)$
W	$\delta(NW, N)$	$\delta(NW, N, NE)$	$\delta(E, W, B)$	$\delta(SE, S, SW)$
NW	$\delta(NW, N)$	$\delta(NW, N, NE)$	$\delta(W, NW, N, NE, E, B)$	U_{dir}
B	N	NE	E	SE

$R_1 \backslash R_2$	S	SW	W	NW	B
N	$\delta(S, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(N, B)$
NE	$\delta(N, NE, E, SE, S, B)$	U_{dir}	$\delta(W, NW, N, NE, E, B)$	$\delta(NW, N, NE)$	$\delta(B, N, NE, E)$
E	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(W, E, B)$	$\delta(NW, N, NE)$	$\delta(E, B)$
SE	$\delta(SE, S)$	$\delta(SE, S, SW)$	$\delta(E, SE, S, SW, W, B)$	U_{dir}	$\delta(B, S, E, SE)$
S	S	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(S, B)$
SW	$\delta(S, SW)$	SW	$\delta(SW, W)$	$\delta(SW, W, NW)$	$\delta(B, S, SW, W)$
W	$\delta(S, SW)$	SW	W	NW	$\delta(W, B)$
NW	$\delta(S, SW, W, NW, N, B)$	$\delta(SW, W, NW)$	$\delta(W, NW)$	NW	$\delta(B, W, NW, N)$
B	S	SW	W	NW	B

Fig. 5. The composition $R_1 \circ R_2$ of atomic relations R_1 and R_2

We now turn our attention to the composition of an atomic with a basic cardinal direction relation. We will first need a few definitions and lemmas before we present the main theorem (Theorem 1).

Definition 3. *A basic cardinal direction relation R is called* rectangular *iff there exist two rectangles (with sides parallel to the x- and y-axes) a and b such that $a\ R\ b$ is satisfied; otherwise it is called* non-rectangular.

Example 2. All atomic relations are rectangular. Relations $B{:}N$ and $B{:}S{:}SW{:}W$ are rectangular while relations $B{:}S{:}SW$ and $B{:}S{:}N{:}SE$ are non-rectangular.

The set of rectangular cardinal direction relations contains the following 36 relations:

$$\begin{gathered}\{B,\ S,\ SW,\ W,\ NW,\ N,\ NE,\ E,\ SE,\\ S{:}SW,\ B{:}W,\ NW{:}N,\ N{:}NE,\ B{:}E,\ S{:}SE,\ SW{:}W,\ B{:}S,\ E{:}SE,\ W{:}NW,\ B{:}N,\ NE{:}E,\\ S{:}SW{:}SE,\ NW{:}N{:}NE,\ B{:}W{:}E,\ B{:}S{:}N,\ SW{:}W{:}NW,\ NE{:}E{:}SE,\\ B{:}S{:}SW{:}W,\ B{:}W{:}NW{:}N,\ B{:}S{:}E{:}SE,\ B{:}N{:}NE{:}E,\\ B{:}S{:}SW{:}W{:}NW{:}N,\ B{:}S{:}N{:}NE{:}E{:}SE,\ B{:}S{:}SW{:}W{:}E{:}SE,\ B{:}W{:}NW{:}N{:}NE{:}E,\\ B{:}S{:}SW{:}W{:}NW{:}N{:}NE{:}E{:}SE\}.\end{gathered}$$

Definition 4. *Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ and $R_2 = R_{21}{:}\cdots{:}R_{2l}$ be two cardinal direction relations. R_1* includes *R_2 iff $\{R_{21}, \ldots, R_{2l}\} \subseteq \{R_{11}, \ldots, R_{1k}\}$ holds.*

Example 3. The basic cardinal direction relation $B{:}S{:}SW{:}W$ includes relation $B{:}S{:}SW$.

Definition 5. *Let R be a basic cardinal direction relation. The* bounding relation *of R, denoted by $\mathcal{B}r(R)$ is the smallest rectangular relation (with respect to the number of tiles) that includes R.*

Example 4. The bounding relation of the basic cardinal direction relation $B{:}S{:}SW$ is relation $B{:}S{:}SW{:}W$.

Definition 6. *Let R be a rectangular cardinal direction relation. We will denote the rectangular relation formed by the westernmost tiles of a relation R by $\mathcal{M}ost(W, R)$. Similarly, we can define the rectangular relations $\mathcal{M}ost(S, R)$, $\mathcal{M}ost(N, R)$ and $\mathcal{M}ost(E, R)$. Moreover, we will denote the atomic relation formed by the southwesternmost tiles of a relation R by $\mathcal{M}ost(SW, R)$. Similarly, we can define the atomic relations $\mathcal{M}ost(SE, R)$, $\mathcal{M}ost(NW, R)$ and $\mathcal{M}ost(NE, R)$. Finally, as a special case, we define $\mathcal{M}ost(B, R) = R$.*

Example 5. Let us consider the rectangular relation $B{:}S{:}SW{:}W$. Then according to Definition 6 we have:

$$\begin{array}{ll} \mathcal{M}ost(W, B{:}S{:}SW{:}W) = SW{:}W, & \mathcal{M}ost(SE, B{:}S{:}SW{:}W) = S, \\ \mathcal{M}ost(S, B{:}S{:}SW{:}W) = S{:}SW, & \mathcal{M}ost(SW, B{:}S{:}SW{:}W) = SW, \\ \mathcal{M}ost(E, B{:}S{:}SW{:}W) = B{:}S, & \mathcal{M}ost(NW, B{:}S{:}SW{:}W) = W, \\ \mathcal{M}ost(N, B{:}S{:}SW{:}W) = B{:}W, & \mathcal{M}ost(NE, B{:}S{:}SW{:}W) = B, \\ \multicolumn{2}{c}{\mathcal{M}ost(B, B{:}S{:}SW{:}W) = B{:}S{:}SW{:}W.} \end{array}$$

The following lemma expresses an important property of operator $\mathcal{M}ost$.

Lemma 1. *Let R_1 be an atomic and R_2 be a rectangular cardinal direction relation. Assume that relation $\mathcal{M}ost(R_1, R_2)$ is $Q_1{:}\cdots{:}Q_t$. Then, the composition of R_1 with $\mathcal{M}ost(R_1, R_2)$ can be computed using formula $R_1 \circ \mathcal{M}ost(R_1, R_2) = \delta(R_1 \circ Q_1, \ldots, R_1 \circ Q_t)$.*

Now, after all the necessary definitions and lemmas, we can present our result.

Theorem 1. *Let R_1 be an atomic cardinal direction relation and R_2 be a basic cardinal direction relation. Then*

$$R_1 \circ R_2 = R_1 \circ \mathcal{M}ost(R_1, \mathcal{B}r(R_2)).$$

The above theorem give us a method to compute the composition $R_1 \circ R_2$ of an atomic cardinal direction relation R_1 with a basic cardinal direction relation R_2. First we have to calculate the relation $\mathcal{M}ost(R_1, \mathcal{B}r(R_2))$. Then we use Lemma 1 and the table of Figure 5 to compute $R_1 \circ R_2$.

We illustrate the above procedure in the following example.

Example 6. Let $R_1 = W$ be an atomic and $R_2 = B{:}S{:}SW$ be a basic cardinal direction relation. Then $\mathcal{M}ost(W, \mathcal{B}r(B{:}S{:}SW)) = SW{:}W$. Thus using Theorem 1, we have $W \circ B{:}S{:}SW{:}W = W \circ SW{:}W$. Using Lemma 1, we also have $W \circ B{:}S{:}SW = \delta(W \circ SW, W \circ W)$. Moreover, using the table of Figure 5 we equivalently have: $W \circ B{:}S{:}SW = \delta(SW, W)$. Finally, expanding operator δ we have:

$$W \circ B{:}S{:}SW = \{SW,\ W,\ SW{:}W\}.$$

The above equation can be easily verified (see also Figure 6).

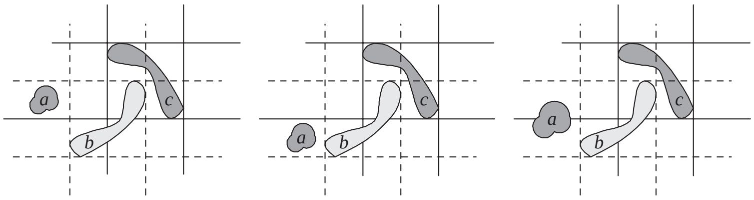

Fig. 6. Composing an atomic with a basic cardinal direction relation

4 Composing a Rectangular with a Basic Cardinal Direction Relation

In this section we will study the composition of a rectangular with a basic cardinal direction relation. We will need the following definition.

Definition 7. *Let R_1 and R_2 be two basic cardinal direction relations. The* tile-union *of R_1 and R_2, denoted by tile-union(R_1, R_2), is a relation formed from the union of tiles of R_1 and R_2.*

For instance, if $R_1 = B{:}S{:}SW$ and $R_2 = S{:}SW{:}W$ then *tile-union*$(R_1, R_2) = B{:}S{:}SW{:}W$. Note that the result of *tile-union* is not always a valid cardinal direction relation. For instance, if $R_1 = W$ and $R_2 = E$ then *tile-union*$(R_1, R_2) = W{:}E \notin \mathcal{D}$.

Theorem 2. *Let $R_1 = R_{11}{:}\cdots{:}R_{1k}$ be a rectangular and R_2 be a basic cardinal direction relation, where $R_{11}, \ldots, R_{1k}$ are atomic cardinal direction relations. Then*

$$R_1 \circ R_2 = \{Q \in \mathcal{D} : Q = \textit{tile-union}(s_1, \ldots, s_k) \wedge s_1 \in R_{11} \circ R_2 \wedge \cdots \wedge s_k \in R_{1k} \circ R_2\}.$$

Using Theorem 2 we can easily derive Algorithm Compose-Rect-Basic that computes the composition R_3 of a rectangular cardinal direction relation R_1 with a basic cardinal direction relation R_2. Assume that R_1 is $R_{11}{:}\cdots{:}R_{1k}$, where $R_{11}, \ldots, R_{1k}$ are atomic cardinal direction relations. Algorithm Compose-Rect-Basic proceed as follows. Initially the algorithm calculates relations S_i, $1 \leq i \leq k$ as the composition of the atomic relation R_{1i} with the basic cardinal direction relation R_2 (as in Section 2). Subsequently, Algorithm Compose-Rect-Basic forms relations by taking the *tile-union* of an atomic cardinal direction relation s_i, from every cardinal direction relation S_i $(1 \leq i \leq k)$. Finally, the algorithm checks whether the result of the union corresponds to a valid cardinal direction relation. If it does then this relation is added to the result R_3; otherwise it is discarded.

We have implemented Algorithm Compose-Rect-Basic and generated the compositions $R_1 \circ R_2$ for every rectangular cardinal direction relation R_1 and basic cardinal direction relation R_2. The results and the code are available from the authors.

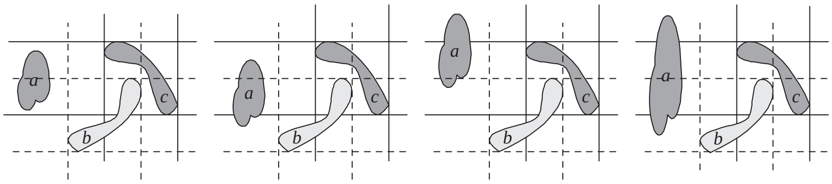

Fig. 7. Composing a rectangular with a basic cardinal direction relation

The following is an example of Algorithm COMPOSE-RECT-BASIC in operation.

Example 7. Assume that we want to calculate the composition of the rectangular relation $W{:}NW$ with the basic relation $B{:}S{:}SW$ (Figure 7). We have:

$$S_1 = W \circ B{:}S{:}SW = \delta(W \circ SW, W \circ SW) = \delta(W, SW) = \{SW, W, SW{:}W\}$$
$$S_2 = NW \circ B{:}S{:}SW = NW \circ W = \{W, NW, W{:}NW\}.$$

Now we construct all relations formed by the union of one relation from S_1 and one relation from S_2. These relations are: $SW{:}W$, $SW{:}NW$, $SW{:}W{:}NW$, W, $W{:}NW$, $W{:}NW$, $SW{:}W$, $SW{:}W{:}NW$ and $SW{:}W{:}NW$. From the above relations only $SW{:}W$, $SW{:}W{:}NW$, W, $W{:}NW$ are valid cardinal direction relations. Therefore, we have:

$$W{:}NW \circ B{:}S{:}SW = \{W,\ SW{:}W,\ W{:}NW,\ SW{:}W{:}NW\}.$$

The above equation can be easily verified (see also Figure 7).

5 Composing Basic Cardinal Direction Relations

Let us now consider the general question of composing two *non-rectangular* basic cardinal direction relations. For this case we have a very interesting result: the language of cardinal direction constraints (as defined in Section 2) *is not expressive enough* to capture the binary relation which is the result of the composition of non-rectangular basic cardinal direction relations. This is illustrated by the following example.

Example 8. Let us consider region variables a, b, c and cardinal direction constraints $a\ S{:}SW{:}W\ b$ and $b\ SW\ c$ (see Figure 8a). The only cardinal direction constraint implied by these two constraints is $a\ SW\ c$. Thus, someone would be tempted to conclude that $(S{:}SW{:}W \circ SW) = SW$. If this equality was correct then for each pair of regions a_0, c_0 such that $a_0\ SW\ c_0$, there exists a region b_0 such that $a_0\ S{:}SW{:}W\ b_0$ and $b_0\ SW\ c_0$. However Figure 8b shows two such regions a_0 and c_0 such that $a_0\ SW\ c_0$ and it is impossible to find a region b_0 such that $a_0\ S{:}SW{:}W\ b_0$.

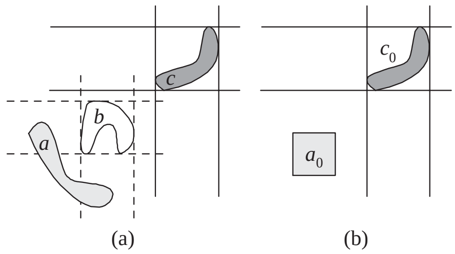

Fig. 8. Illustration of Example 8

If we consider this example carefully, we will notice that the given constraint on a and b implies the following constraint on a: the *area* covered by each region substituted for a cannot be rectangular; it should extend so that it covers tiles $S(b), SW(b)$ and $W(b)$ for any region b. Obviously this constraint is not expressible in the language of cardinal direction relations presented in Section 2.

It is an open question to define an appropriate set of predicates that could be used to augment the constraint language of Section 2 so that the constraints needed to define the result of a composition operation are expressible in all cases.

It is important to point out that the above non-expressibility result should not deter spatial database practitioners who would like to consider adding the cardinal direction relations described in this paper to their system. The discussion of the introduction (i.e., using the inferences of a composition table for spatial relations in order to prune the search space during optimisation of certain queries) still applies but now one has to be careful to say that she is using a constraint propagation mechanism and not a composition table!

Unfortunately, we cannot be as positive about using the cardinal direction relations defined in this paper in the constraint databases frameworks of [8] or [9]. In these frameworks, the class of constraints involved must be closed under the operation of *variable elimination*. Example 8 above demonstrates that this is not true for the class of cardinal direction constraints examined in this paper. For example, if we have constraints

$$a \; S{:}SW{:}W \; b, \quad b \; SW \; c$$

and we eliminate variable b, the result of the elimination is not expressible in the constraint language we started with! So the language of Section 2 needs to be modified in order to be used in a constraint database model. We are currently working on extending this language to remove this limitation.

6 Conclusions

In this paper we gave a formal presentation of the cardinal direction model of Goyal and Egenhofer [6]. We used our formal framework to study the composition operation for cardinal direction relations in this model. We considered progressively more expressive classes of cardinal direction relations and gave composition algorithms for these classes. Finally, we have demonstrated that in some cases, the binary relation resulting from the composition of two cardinal direction relations cannot even be expressed using the vocabulary defined in [6].

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